Remark: All vector spaces in these questions are finite dimensional over a common field. Each question carries 20 marks.

1. Show that all the bases in a vector space have the same size.

Solution: Let V be a finite dimensional vector space in which $A = \{v_1, v_2, \cdots, v_m\}$ and $B = \{w_1, w_2, \cdots, w_n\}$, where $m, n \in \mathbb{N}$, be two sets of vectors. First we show that if A is linearly independent and B spans V, then $m \leq n$.
Let $m > n$. Now, A is linearly independent $\implies v_i \neq 0, \forall\ i \in \{1, \cdots, m\}$. Also B spans $V \implies v_1 = \sum_{i=1}^n \alpha_i w_i$. Then $\alpha_i \neq 0$, for some $i \in \{1, \cdots, n\}$, otherwise $v_1 = 0$. Let k be the first index for which $\alpha_k \neq 0$. Now interchange the indices $k \leftrightarrow 1$ in B we get $v_1 = \sum_{i=1}^n \alpha_i w_i$, where $\alpha_1 \neq 0$ $\implies w_1 = \frac{1}{\alpha_1}\Big(v_1 - \sum_{i=2}^n \alpha_i w_i\Big)$. Therefore if $B_1 = \{v_1, w_2, \cdots, w_n\}$, then B_1 also spans V. Thus, $v_2 = \beta_1 v_1 + \sum_{i=2}^n \beta_i w_i$. Now if $\beta_i = 0, \forall\ i \in \{2, \cdots, n\} \implies v_2 = \beta v_1 \implies v_1, v_2$ are linearly dependent, a contradiction, hence $\beta_i \neq 0$, for some $i \in \{2, \cdots, n\}$. Let j be the first indices in $\{2, \cdots, n\}$ for which $\beta_j \neq 0$. Similarly as above by rewriting the indices in B_1 we get $\beta_2 \neq 0$. Hence $v_2 = \frac{1}{\beta_2}\Big(v_2 - \beta_1 v_1 - \sum_{i=3}^n \beta_i w_i\Big)$, implies $B_2 = \{v_1, v_2, w_3, \cdots, w_n\}$ spans V.
Since $m > n$ we can continue this process up to n times. At the n-th step we get $B_n = \{v_1, v_2, \cdots, v_n\}$ spans V, implies $A = \{v_1, \cdots, v_m\}$ is linearly dependent, a contradiction. Hence $m \leq n$.
Now, if A and B are both bases of V then using above result we get $m \leq n$ and $n \leq m \implies m = n$. Hence all the bases of a finite dimensional vector space have the same size.

2. If U is a vector subspace of a vector space V, then show that $dim(U) + dim(U^0) = dim(V)$.

Solution: As U is a subspace of V, $U + U^0 = V$ and $U \cap U^0 = \{0\}$, where U^0 is the complementary subspace of U in V. Let $dim(U) = m$, $dim(U^0) = n$ and $\{v_1, \cdots, v_m\}$, $\{w_1, \cdots, w_n\}$ are the bases of U and U^0 respectively.
Then for any $x \in V$, $x = u + v$ for some $u \in U$ and $v \in U^0$, implies $x = \sum_{i=1}^m \alpha_i v_i + \sum_{j=1}^n \beta_j w_j$, implies $B = \{v_1, \cdots, v_m, w_1, \cdots, w_n\}$ spans V. Also if $\sum_{i=1}^m \alpha_i v_i + \sum_{j=1}^n \beta_j w_j = 0 \implies \sum_{i=1}^m \alpha_i v_i = -\sum_{j=1}^n \beta_j w_j \in U \cap U^0 = \{0\} \implies \alpha_i = 0,\ \beta_j = 0,\ \forall\ i \in \{1, \cdots, m\},\ j \in \{1, \cdots, n\} \implies B$ is linearly independent. Thus B is a basis of V which implies that $dim(V) = m + n = dim(U) + dim(U^0)$.

3. If $T : V \to W$ is a linear transformation between vector spaces then show that $rank(T) + nullity(T) = dim(V)$, conclude that if $dim(V) = dim(W)$, then T is injective iff T is surjective.

Solution: Let $\{v_1, \cdots, v_m\}$ be a basis of null space of T which is a subspace of V. Then, it can be extended to a basis of V. Let $\{v_1, \cdots, v_m, v_{m+1}, \cdots, v_n\}$ be the corresponding basis of V, where $n = dim(V) \geq m = nullity(T)$. Therefore, if we show that $\{T(v_{m+1}), \cdots, T(v_n)\}$ is a basis for range of T, then $rank(T) + nullity(T) = (n - m) + m = n = dim(V)$.
Let $x \in$ range of $T \implies x = T(v)$, for some $v \in V \implies x = T(\sum_{i=1}^n \alpha_i v_i) = T(\sum_{i=m+1}^n \alpha_i v_i)$, as $T(v_i) = 0,\ \forall\ i \in \{1, \cdots, m\}$, implies $\{T(v_{m+1}), \cdots, T(v_n)\}$ spans the range of T. Now, $\sum_{i=m+1}^n \alpha_i T(v_i) = 0 \implies T(\sum_{i=m+1}^n \alpha_i v_i) = 0 \implies \sum_{i=m+1}^n \alpha_i v_i \in$ null space of $T \implies \sum_{i=m+1}^n \alpha_i v_i = \sum_{j=1}^m \beta_j v_j \implies \sum_{j=1}^m \beta_j v_j - \sum_{i=m+1}^n \alpha_i v_i = 0 \implies \beta_j = \alpha_i = 0, \forall\ j \in \{1, \cdots, m\}, i \in \{m+1, \cdots, n\}$, as $\{v_1, \cdots, v_n\}$ is a basis of V. Therefore, $\{T(v_{m+1}), \cdots, T(v_n)\}$ is linearly independent and thus a basis for range of T.
Now if $dim(V) = dim(W)$, then $rank(T) + nullity(T) = dim(V) = dim(W)$. Therefore if T is injective $\Leftrightarrow$ null space of $T = \{0\} \Leftrightarrow nullity(T) = 0 \Leftrightarrow rank(T) = dim(W) \Leftrightarrow$ range of $T = W$ (as range of T is a

subspace of W) $\Leftrightarrow T$ is surjective.

3. Let $x_i, y_i, 1 \leq i \leq m$. be $2m$ vectors in an inner product space V such that $< x_i, x_j >=< y_i, y_j >$ for all i, j $(1 \leq i, j \leq m)$. Then show that there is an orthogonal linear transformation $T : V \to V$ such that $Tx_i = y_i, 1 \leq i \leq m$.

Solution: Let $X = span(\{x_1, \cdots, x_m\})$ and $Y = span(\{y_1, \cdots, y_m\})$. Now if $< x_i, x_j >=< y_i, y_j >, \forall\ i, j \in \{1, \cdots, m\}$, then $\sum_{i=1}^{k} \alpha_i y_i = 0 \Leftrightarrow < \sum_{i=1}^{k} \alpha_i y_i, y_j >= 0, \forall\ j \in \{1, \cdots, k\} \Leftrightarrow \sum_{i=1}^{k} \alpha_i < y_i, y_j >= 0 \Leftrightarrow \sum_{i=1}^{k} \alpha_i < x_i, x_j >= 0 \Leftrightarrow < \sum_{i=1}^{k} \alpha_i x_i, x_j >= 0, \forall\ j \in \{1, \cdots, k\} \Leftrightarrow \sum_{i=1}^{k+1} \alpha_i x_i = 0$. Thus, $\{x_1, \cdots, x_k\}$ is linearly independent iff $\{y_1, \cdots, y_k\}$ is linearly independent. Therefore, $dim(X) = dim(Y)$ and $B_X = \{x_1, \cdots, x_k\}$ is a basis of X iff $B_Y = \{y_1, \cdots, y_k\}$ is a basis of Y.
Now we show that B_X and B_Y can be extended to bases of V, say $B_1 = \{x_1, \cdots, x_n\}$ and $B_2 = \{y_1, \cdots, y_n\}$, such that $x_j \perp span(\{x_1, \cdots, x_{j-1}\})$, $y_j \perp span(\{y_1, \cdots, y_{j-1}\}), \forall\ j \in \{k+1, \cdots, n\}$ and $< x_i, x_j >=< y_i, y_j >, \forall\ i, j \in \{1, \cdots, n\}$, where $n = dim(V) \geq k$.
Since X and Y are subspaces of V, then $V = X \bigoplus X^{\perp} = Y \bigoplus Y^{\perp}$. Choose $x_{k+1} \in X^{\perp}$ and $y_{k+1} \in Y^{\perp}$ such that $< x_{k+1}, x_{k+1} >=< y_{k+1}, y_{k+1} >$. Let $X_{k+1} = span(X \bigcup \{x_{k+1}\})$ and $Y_{k+1} = span(Y \bigcup \{y_{k+1}\})$. Then X_{k+1} and Y_{k+1} are subspaces of V and $dim(X_{k+1}) = dim(Y_{k+1})$. Also $\{x_1, \cdots, x_{k+1}\}$ and $\{y_1, \cdots, y_{k+1}\}$ are bases of X_{k+1} and Y_{k+1} respectively, such that $x_{k+1} \perp \{x_1, \cdots, x_k\}$ and $y_{k+1} \perp \{y_1, \cdots, y_k\}$ and $< x_i, x_j >=< y_i, y_j >, \forall\ i, j \in \{1, \cdots, k+1\}$.
Again, since X_{k+1} and Y_{k+1} are subspaces of V and $dim(X_{k+1}) = dim(Y_{k+1})$, then $V = X_{k+1} \bigoplus X_{k+1}^{\perp} = Y_{k+1} \bigoplus Y_{k+1}^{\perp}$. Similarly as above we can choose $x_{k+2} \in X_{k+1}^{\perp}$ and $y_{k+2} \in Y_{k+1}^{\perp}$ such that $< x_{k+2}, x_{k+2} >=< y_{k+2}, y_{k+2} >$. Then $\{x_1, \cdots, x_{k+2}\}$ and $\{y_1, \cdots, y_{k+2}\}$ are bases of $X_{k+2} = span(X_{k+1} \bigcup \{x_{k+2}\}) \subset V$ and $Y_{k+2} = span(Y_{k+1} \bigcup \{y_{k+2}\}) \subset V$ respectively, such that $x_{k+2} \perp \{x_1, \cdots, x_{k+1}\}$, $y_{k+2} \perp \{y_1, \cdots, y_{k+1}\}$ and $< x_i, x_j >=< y_i, y_j >, \forall\ i, j \in \{1, \cdots, k+2\}$. Since $n = dim(V)$, by continuing this process after $n-k$ step we get the desired bases B_1 and B_2 of V.
Now, consider a map $T : V \to V$ such that $T\Big(\sum_{i=1}^{n} a_i x_i\Big) = \sum_{i=1}^{n} a_i y_i$, for all scalar $a_i, i \in \{1, \cdots, n\}$. Then T is linear, as $T(\alpha v + \beta w) = T\Big(\alpha \sum_{i=1}^{n} a_i v_i + \beta \sum_{i=1}^{n} b_i v_i\Big) = T\Big(\sum_{i=1}^{n} (\alpha a_i + \beta b_i) v_i\Big) = \sum_{i=1}^{n} (\alpha a_i + \beta b_i) w_i = \alpha T(v) + \beta T(w)$, and $T(x_i) = y_i, \forall\ i \in \{1, \cdots, n\}$. Now, since $< x_i, x_j >=< y_i, y_j >, \forall\ i, j \in \{1, \cdots, n\}$, then for any $v = \sum_{i=1}^{n} \alpha_i x_i$ and $w = \sum_{i=1}^{n} \beta_i x_i$ in V,
$< Tx, Ty >=< T(\sum_{i=1}^{n} \alpha_i x_i), T(\sum_{i=1}^{n} \beta_i x_i) >=< \sum_{i=1}^{n} \alpha_i y_i, \sum_{i=1}^{n} \beta_i y_i >= \sum_{i=1}^{n} \sum_{j=1}^{n} \alpha_i \bar{\beta}_j < y_i, y_j >= \sum_{i=1}^{n} \sum_{j=1}^{n} \alpha_i \bar{\beta}_j < x_i, x_j >=< \sum_{i=1}^{n} \alpha_i x_i, \sum_{i=1}^{n} \beta_i x_i >=< x, y >$. Hence T is orthogonal.

5. Let $L(U, V)$ denote the set of all linear transformation from the vector space U to the vector space V. Thus, $L(U, V)$ is a vector space with point wise operation. Show that its dimension is the product of the dimensions of U and V.

Solution: Let $B_U = \{u_1, \cdots, u_n\}$ and $B_V = \{v_1, \cdots, v_m\}$ be the bases for U and V respectively. For each pair (p, q), where $1 \leq p \leq m$ and $1 \leq q \leq n$, consider a map $T_{pq} : U \to V$ such that $T_{pq}(\sum_{i=1}^{n} a_i u_i) = a_q v_p$ for all scalar $a_i, i \in \{1, \cdots, n\}$. Then T_{pq} is linear as $T_{pq}(\alpha v + \beta w) = T_{pq}\Big(\alpha \sum_{i=1}^{n} a_i v_i + \beta \sum_{i=1}^{n} b_i v_i\Big) = T_{pq}\Big(\sum_{i=1}^{n} (\alpha a_i + \beta b_i) v_i\Big) = (\alpha a_q + \beta b_q) w_p = \alpha T(v) + \beta T(w)$. Hence $T_{pq} \in L(U, V), \forall p \in \{1, \cdots, m\}, q \in \{1, \cdots, n\}$. Now, if we show that $B_{L(U,V)} = \{T_{pq}; 1 \leq p \leq m, 1 \leq q \leq n\}$ forms a basis for $L(U, V)$, then $dim(L(U, V)) = nm = dim(U) \times dim(V)$.
Let $T \in L(U, V)$. Then for each $j \in \{1, \cdots, n\}$, $T(u_j) \in V$. Let $T(u_j) = \sum_{p=1}^{m} a_{pj} v_p$, for some scalars $a_{pj}, 1 \leq p \leq m$. We show that $T = \sum_{p=1}^{m} \sum_{q=1}^{n} a_{pq} T_{pq} \implies B_{L(U,V)}$ spans $L(U, V)$.
Since, $\sum_{p=1}^{m} \sum_{q=1}^{n} a_{pq} T_{pq}(u_j) = \sum_{p=1}^{m} a_{pj} v_p = T(u_j)$, for all $j \in \{1 \cdots, n\}$, $\implies T = \sum_{p=1}^{m} \sum_{q=1}^{n} a_{pq} T_{pq}$. Also $\sum_{p=1}^{m} \sum_{q=1}^{n} a_{pq} T_{pq} = 0 \implies \sum_{p=1}^{m} \sum_{q=1}^{n} a_{pq} T_{pq}(u_j) = 0, \forall j \in \{1 \cdots, n\} \implies \sum_{p=1}^{m} a_{pj} v_p = 0 \implies a_{pj} = 0, \forall\ p \in \{1, \cdots, m\}, j \in \{1, \cdots, n\}$. Thus $B_{L(U,V)}$ is linearly independent. Hence $B_{L(U,V)}$ is a basis for $L(U, V)$.